CONTINUOUS FIBER COMPOSITE STRENGTH

Composite stress is given by the simple rule of mixtures:

\[ \sigma_1 = V_f \sigma_f + (1 - V_f) \sigma_m \]

for strains before onset of matrix cracking.

Two Cases
Case 1: Fiber has a higher strain to failure, and
Case 2: Matrix has the higher strain to failure

In the figure above (Hull, D. and Clyne, T. W. 1996. An Introduction to Composite Materials. Cambridge Univ. Press):

\[ \sigma_{1u} = V_f \sigma_{f_{mu}} + (1 - V_f) \sigma_{m_{fu}} \]

\[ V_f' = f' = \frac{\sigma_{mu}}{\sigma_{fu} - \sigma_{f_{mu}} + \sigma_{mu}} \]

\[ \sigma_{1u} = V_f \sigma_{f_{mu}} + (1 - V_f) \sigma_{m_{fu}} \]

\[ V_f' = f' = \frac{\sigma_{mu} - \sigma_{m_{fu}}}{\sigma_{fu} - \sigma_{m_{fu}} + \sigma_{mu}} \]
SHORT FIBER COMPOSITES

Cost effective manufacturing often drives the use of discontinuous or “short” reinforcing elements in composites. These materials differ from continuous reinforced composites because the fibers do not extend throughout the entire material. Therefore, load is not directly applied to each reinforcing element.

Stress Transfer
The most commonly quoted theory of stress transfer in discontinuous element composites is the shear-lag analysis.\(^1\) Classically, this element is considered as a discontinuous cylindrical fiber embedded in a continuous matrix. Differences in longitudinal strain in the matrix and adjacent fiber (matrix >> fiber) will result in shear stresses at the interface. Tensile load is transferred to the fibers by a shearing mechanism between fibers and matrix. The stresses acting on this body are:

\[
\begin{align*}
\sigma_c & = \sigma_f + d\sigma_f \\
\tau & = \frac{1}{2} \left( \frac{\sigma_f + d\sigma_f}{r} \right) dz
\end{align*}
\]

Ignoring stress transfer at the fiber end cross sections and interaction between neighboring fibers, we can calculate normal stress distribution by a simple equilibrium analysis. Consider the force equilibrium on the infinitesimal length, \(dz\):

\[
\left( \pi r^2 \right) \sigma_f + \left( 2\pi r dz \right) \tau = \left( \pi r^2 \right) \left( \sigma_f + d\sigma_f \right)
\]

Reducing this relation, we find that the change in fiber stress with length is:

\[ \frac{d\sigma_f}{dz} = \frac{2\tau}{r} \]

This relation implies that the change in fiber stress is proportional to the shear stress at the interface between the fiber and matrix. The fiber stress at any point along the fiber is obtained by integrating this equation with respect to \( z \).

\[ \sigma_f = \sigma_{f0} + \frac{2}{r} \int_0^z \tau \, dz \]

The stress at the fiber end, \( \sigma_{f0} \), is often negligible and can be neglected in the formulation:

\[ \sigma_f \approx \frac{2}{r} \int_0^z \tau \, dz \]

This formulation requires knowledge for the shear stresses along the fiber length. Although these are seldom known, one frequently used assumption is that the matrix material is perfectly rigid and perfectly plastic. In this case, the shear stress would develop instantly at the end of the fiber, reaching a maximum of the yield stress (\( \tau_y \)). The normal stresses in the fiber can then be given as:

\[ \sigma_f = \frac{2\tau_y z}{r} \]

The maximum normal stress in the fiber will occur at mid-length (\( z = l/2 \)):

\[ \sigma_{f,\text{max}} = \frac{\tau_y l}{r} \]

Assuming continuity of strains between the fiber, matrix, and composite (i.e. \( \varepsilon_f = \varepsilon_m = \varepsilon_c \)), the maximum fiber stress can be given as:

\[ \sigma_{f,\text{max}} = \frac{E_f}{E_c} \sigma_c \]

Because the normal stress in the fiber changes along the fiber length, we can assume that a minimum fiber length (\( l_t \)) exists to transfer the maximum fiber stress. This minimum fiber length can be calculated by combining the previous two equations and rearranging:

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Short-Fiber Composites

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If we consider that the maximum fiber stress can be equated to that required to fail the fiber ($\sigma_{fu}$), then we can imagine that extremely short fibers will never develop enough stress to fail. This concept leads to the theory that a critical fiber length ($l_c$) exists for effective reinforcement:

$$\frac{l_c}{d} = \frac{\sigma_{max}}{2\tau_y} = \frac{(E_f / E_c)\sigma_c}{2\tau_y}$$

Semi-Empirical Equations

Halpin-Tsai Equations

Halpin and Tsai\(^2\) performed a more exact micro-mechanics analysis on unidirectional composites with short fibers. For simplicity, approximate equations were produced that afford a more precise prediction of properties.

The basic form of the relations is given as:

$$\frac{E_i}{E_m} = \frac{1 + \eta_i \xi_i \nu_f}{1 - \eta_i \nu_f}$$

where: $\eta_i = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi_i}$

$i$ indicates the two principal material directions; longitudinal (L) or transverse (T) to the fiber direction.

The reinforcement shape parameter, $\xi_i$ depends on the direction of loading and the filler shape:

$$\xi_L = 2 \frac{L}{x}, \quad \xi_T = 2 \quad \text{where: } x \text{ equals the fiber diameter or platelet thickness}$$

For random filler orientation the modulus is approximated as\(^3\):

$$E_r = \frac{3E_L}{8} + \frac{5E_T}{8} \quad \text{OR} \quad G_r = \frac{E_L}{8} + \frac{E_T}{4}$$


Short-Fiber Composites

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Discontinuous Fiber Composite Strength

Assuming (1) an aligned short fiber composite with (2) the load placed parallel to the fiber direction, the overall stress in the composite can be calculated as:

\[ \sigma_c = \bar{\sigma}_f \nu_f + \sigma_m \nu_m \]

And peak stress at the center of fiber = \( \sigma_f^\text{max} = \frac{2\tau l}{d} \) because force acting on half a fiber is:

Assuming a linear stress variation in the fiber

\[ F_1 = \sigma_f \pi D^2 / 4 \]
\[ F_2 = [\sigma_f + d\sigma_f] \pi D^2 / 4 \]
\[ F_3 = \tau \pi D dx \]

For equilibrium, \( F_1 + F_3 = F_2 \) and by integration and simplification, we get

\[ \sigma_f = 4\tau (l/2 - x) / D \]

Note, \( x \) is measured from the mid-length of the fiber and \( \tau \) is assumed to be a constant.

In solving these equations, recall that the normal stress in a fiber changes along the discontinuous fiber length.
The average fiber stress, $\bar{\sigma}_f$, is determined by taking the area under the stress-fiber length graph and dividing by the fiber length.

Now consider the following three cases:

<table>
<thead>
<tr>
<th>Length</th>
<th>Governing Equation</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l &lt; l_t$</td>
<td>$\sigma_c = \frac{1}{2} \sigma_f^{\max} \nu_f + \sigma_m \nu_m$</td>
<td>Triangular stress over $l_t$</td>
</tr>
<tr>
<td>$l &gt; l_t$</td>
<td>$\sigma_c = (1 - l_t / 2l) \sigma_f^{\max} \nu_f + \sigma_m \nu_m$</td>
<td>$1 - l_t / 2l \approx 1$</td>
</tr>
<tr>
<td>$l &gt;&gt; l_t$</td>
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<td>$1 - l_t / 2l \approx 1$</td>
</tr>
</tbody>
</table>

Considering both the ultimate fiber stress ($\sigma_{fu}$) and matrix stress ($\sigma_{mu}$) we can compute the composite material strength ($\sigma_{cu}$)...
Interphase Volume

For composites with engineered filler-matrix interfaces, a third phase must be considered. The material interphase has properties higher than the matrix and is often included in the fiber volume fraction. Therefore, an effective fiber volume fraction \( \nu_{f}^{\text{eff}} \) and fiber modulus \( E_{f}^{\text{eff}} \) is substituted in the equations above for \( \nu_{f} \) and \( E_{f} \) respectively. These terms can be computed as:

\[
\nu_{f}^{\text{eff}} = \nu_{f} + \nu_{i}
\]

\[
E_{f}^{\text{eff}} = \zeta E_{f} \nu_{f}^{'} + E_{i} \nu_{i}, \text{ where } \nu_{f}^{'} \text{ is the filler volume fraction in the fiber-interphase complex.}
\]

A common simplification assumes:

\[
E_{i} = E_{f} = E_{f}^{\text{eff}}
\]

Using this assumption, the effective fiber volume fraction can be solved by fitting the modified Halpin-Tsai equations to the reinforced composite modulus at varying levels of \( \nu_{f} \) and solving for \( \nu_{i} \)

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Equations Relating to Weight and Volume Fractions

By Definition

Volume Fraction \( \nu_f = \frac{V_f}{V_c} \) and \( \nu_m = \frac{V_m}{V_c} \)

Weight Fraction \( \omega_f = \frac{W_f}{W_c} \) and \( \omega_m = \frac{W_m}{W_c} \)

Invoking the definition of density:

\( \omega_f = \frac{W_f}{W_c} = \frac{\rho_f V_f}{\rho_c V_c} = \frac{\rho_f}{\rho_c} \nu_f \) and \( \omega_m = \frac{W_m}{W_c} = \frac{\rho_m V_m}{\rho_c V_c} = \frac{\rho_m}{\rho_c} \nu_m \)

By identity:

\( \nu_f + \nu_m = \omega_f + \omega_m = 1 \)

Assuming that the material is void free:

\( W_c = W_f + W_m \) and \( V_c = V_f + V_m \)

Invoking the definition of density:

\( \rho_c V_c = \rho_f V_f + \rho_m V_m \)

Dividing each side by the composite’s volume leads to:

\( \rho_c = \rho_f \nu_f + \nu_m \rho_m \)

or using similar paths for weight fraction:

\[ \frac{1}{\rho_c} = \frac{\omega_f}{\rho_f} + \frac{\omega_m}{\rho_m} \]