BENDING OF BEAMS

TO BEGIN LOOKING AT APPLICATIONS OF ELASTICITY TO MORE REAL PROBLEMS LET'S CONSIDER BEAMS.

DEFINE A BEAM AS AN ELEMENT WITH ONE DIMENSION IS SIGNIFICANTLY GREATER THAN THE OTHER TWO.

THEORY OF ELASTICITY (CLASSICAL) SOLUTIONS CAN ONLY YIELD SOLUTIONS TO SIMPLE PROBLEMS UNLESS SIGNIFICANT COMPLEXITY IS FOLLOWED.

HOWEVER, THEORY OF ELASTICITY CAN BE USED TO:

1) PLACE LIMITATIONS ON ELEMENTARY THEORY SOLUTIONS (CLOSED FORM SOLUTIONS WITH SIGNIFICANT ASSUMPTIONS)
2) BE THE BASIS OF APPROXIMATE SOLUTIONS (I.E., FEM, FD, ETC.)
3) PROVIDE EXACT SOLUTIONS FOR SIMPLE CONFIGURATIONS.

EXACT SOLUTIONS (TO SIMPLE PROBLEMS)

ASSUME PRISMATIC CROSS-SECTION AND POSITIVE MOMENTS
Assume \( \sigma_x = ky \)

\[
\begin{align*}
\sigma_y &= \sigma_z = 0, \\
\alpha_y &= \alpha_z = 0, \\
\alpha_{yz} &= \alpha_{zx} = 0
\end{align*}
\]

\( \mathbf{K} = \text{constant} \)

\( \phi = 0 \) @ \( y = 0 \) - neutral axis

Assumed stress distribution must satisfy the boundary conditions

\[
\int_{\Lambda} \sigma_x \, dA = 0
\]

\[
-\int_{\Lambda} \gamma \sigma_x \, dA = M_x
\quad \text{(negative indicates that stress is compressive in positive y direction)}
\]

\[
\int_{\Lambda} \gamma_y \, dA = \int_{\Lambda} \gamma_z \, dA = 0
\]

\[
\int_{\Lambda} \gamma_{xy} \, dA = \int_{\Lambda} \gamma_{yz} \, dA = \int_{\Lambda} \gamma_{zx} \, dA = 0
\]

(No forces applied in x, y, z, or shear planes)
\[ \int_k y \, dA = 0 \quad \Rightarrow -\frac{1}{k} \int_k y^2 \, dA = M_2 \]

\[ \frac{6y^2}{I} = 0 \quad \Rightarrow -\frac{1}{k} \frac{6y^3}{I} = M_2 \]

\[ K = -\frac{M_2}{I} \]

\[ \sigma_x = ky = -\frac{M_2 y}{I} \]

(CLASSICAL FLEXURE STRESS FORMULA)

\[ \sigma_{y_{\max}} = -\frac{M_2 y_{\max}}{I} = \frac{M}{I} \]

\[ \sigma_{\max} = \frac{M}{S} = \frac{M_2}{S} \]

\[ E_{\text{elastic section modulus}} \]

\[ \text{MOR.} = \frac{M}{S_{\text{gross}}} \quad - \text{TABULATED FOR WOOD, COMPOSITES \& FOAMS.} \]

**IS THIS AN EXACT SOLUTION FOR**
1) **POINT LOAD?**
2) **DISTRIBUTED LOAD?**
3) **COUPLE?**
**KINEMATIC RELATIONSHIPS (GEOMETRY OF DEFORMATIONS)**

**ASSUME PLANE SECTIONS REMAIN PLANE**

\[
\varepsilon_x = \frac{\sigma_x}{E} = -\frac{M_y}{EI}
\]

\[
\varepsilon_y = \varepsilon_z = \nu \frac{\sigma_x}{E} = \nu \frac{M_y}{EI}
\]

\[
\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0
\]

*The elongation of the neutral axis is zero for simple supports. Since beam is in pure bending, all sections are subjected to the same bending curvature deformation.*
The curvature of the neutral axis in the xy plane in terms of $v$ is

\[ \frac{1}{r_x} = \left[ 1 + \left( \frac{dv}{dx} \right)^2 \right]^{3/2} \]

For small deformations, $\frac{dv}{dx} \ll 1$

\[ \frac{1}{r_x} \approx \frac{d^2v}{dx^2} \]

Traditional sign convention is for the beam to be concave down (compression on the bottom) for positive $v$.

From figure

\[ \tan(\delta) = \frac{d\delta}{r_x} = \frac{d\delta}{y} = -\frac{\varepsilon_x}{y} ds \]
WHERE $\Delta s = $ arc length mm

For small deformations, $\Delta s = \Delta y \Delta x$

$\theta = \frac{\Delta y}{\Delta x}$ (slope)

$\theta$ increases from left to right

Combining eq's with $\Delta s$

\[
\frac{1}{1_x} = -\frac{\Delta y}{\Delta x} = -\frac{\Delta y}{\Delta x} = -\frac{M_z}{E I_x}
\]

Also, using similar substitutions

\[
\frac{1}{1_z} = -\frac{\Delta x}{\Delta y} = -\frac{\sqrt{M_z}}{E I_z}
\]

Traditional beam bending equation is found by combining eq's with $\Delta$

\[
\frac{d^2 y}{dx^2} = \frac{M_z}{E I_x}
\]

(Bernoulli-Euler law of elementary bending theory)

Reviewing:

\[
\frac{1}{1_z} = -\frac{\sqrt{M_z}}{E I_z}
\]
BENDING OF BEAMS WITH ARBITRARY CROSS SECTION

Following the same procedure as for prismatic sections, assume plane sections remain plane.

\[ \sigma_x = \gamma y + \gamma y_x \varepsilon_x + \gamma_x \varepsilon_x \]

\[ \sigma_y = \varepsilon_x = \gamma y_x = \gamma y_x \varepsilon_x = \gamma x \varepsilon_x = 0 \]

Imposing the boundary conditions:

\[ \int_A \sigma_x \, dA = 0 \quad (\varepsilon_x = 0) \]

\[ \int_A z \sigma_x \, dA = M_y \quad (\varepsilon M_y = 0) \]

\[ \int_A y \sigma_x \, dA = M_x \quad (\varepsilon M_z = 0) \]
Substituting:

\[ S_A (c_1 y + c_2 z + c_3) dA = 0 \]

\[ c_1 S_A y dA + c_2 S_A z dA + c_3 S_A dA = 0 \]

Similarly:

\[ c_1 S_A y^2 dA + c_2 S_A z^2 dA + c_3 S_A z dA = M_y \]

\[ c_1 S_A y^2 dA + c_2 S_A y z dA + c_3 S_A y dA = M_z \]

If the origin for \( y \) and \( z \) axes is to be located at the centroid of the cross section:

\[ S_A y dA = S_A z dA = 0 \]

\[ \therefore c_3 S_A dA = 0 \]

\[ \therefore c_3 = 0 \]

This also indicates that the neutral axis is at the centroid.

Now

\[ S_A z^2 dA = I_y \]

\[ S_A y^2 dA = I_z \]

\[ S_A y z dA = I_{xy} \]

\[ S_A y z dA = I_{xy} \]

Moment of Inertia

Product of Inertia
\[ C_1 I_{yz} + C_2 I_y = M_y \]
\[ C_1 I_z + C_2 I_{yz} = M_z \]
\[ C_1 = \frac{M_z - C_2 I_{yz}}{I_z} \]
\[ \frac{M_z - C_2 I_{yz}}{I_z} I_{yz} + C_2 I_y = M_y \]
\[ M_z I_y - C_2 I_{yz}^2 + C_2 I_y I_z = M_y I_z \]
\[ C_2 (I_y I_z - I_{yz}^2) = M_y I_z - M_z I_{yz} \]
\[ C_2 = \frac{M_y I_z - M_z I_{yz}}{I_y I_z - I_{yz}^2} \]
\[ C_1 = \frac{M_z (I_y I_z - I_{yz}^2) - M_y I_z - M_z I_{yz}}{I_z (I_y I_z - I_{yz}^2)} \]

Substituting into the boundary condition Eqs
\[ \sigma_x = \frac{C_1 I_y I_z + C_2 I_{yz}}{I_y I_z - I_{yz}^2} \]

To find the neutral axis \( C : \)
\[ (M_y I_z + M_z I_{yz}) I_z - (M_y I_{yz} + M_z I_y) I_y = 0 \]

Equation of a line through \( C \)
Determine angle w.r.t. z-axis

\[
\tan \phi = \frac{y}{z} = \frac{My_Iz + Mz_Iy'z}{M_y'z + Mz'Iy}
\]

Maximum stress occurs at point furthest from the centroid.

There is an orientation where \( I_y'z = 0 \)

\[
\sigma_x = \frac{My'}{I_y'} - \frac{Mz'}{I_z'}
\]

\( y'z' \) are principle axis.

Since \( I \) is second order tensor can use Mohr's circle

\[
I_y' = \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2} \cos \phi - I_y' \sin \phi
\]

\[
\tan 2\theta = -\frac{2I_{yz}}{I_y - I_z}
\]

The principal moment of inertia are

\[
I_{1,2} = \frac{I_y + I_z}{2} + \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2}
\]
BENDING OF A CANTILEVER

\[ z \ll 2h \] (i.e., thin section)

Because the section is thin, we can assume a plane stress problem.

i.e., \( \sigma \neq f(z) \)

From previous work:

\[ \sigma_x = -\frac{P}{I} y \]

\[ \sigma_y = 0 \]

\[ \tau_{xy} = -\frac{P}{2I} (h^2 - y^2) \]

To derive deflection equations, we must use Hooke's law to relate stress \& strain:

\[ \frac{\partial u}{\partial x} = -\frac{Pxy}{EI} \]

\[ \frac{\partial v}{\partial y} = \frac{vPxy}{EI} \]

\[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1+v)}{E} \tau_{xy} = -\frac{(1+v)}{EI} \frac{P}{(h^2 - y^2)} \]
INTEGRATING THE NORMAL STRESS EQUATIONS

\[ u = \int \frac{2u}{\partial x} \, dx = \int \frac{Py}{EI} \, dx \]

\[ u = -\frac{Py^2}{2EI} + u_i(x) \]

\[ v = \int \frac{2v}{\partial y} \, dy = \int \frac{Vp}{EI} \, dy \]

\[ v = \frac{Vp}{2EI} + v_i(x) \]

NOW DIFFERENTIATE THE DISPLACEMENT EQUATIONS (\(u\) WITH RESPECT TO \(y\) AND \(v\) WITH RESPECT TO \(x\))

\[ \frac{\partial u}{\partial y} = -\frac{Px^2}{2EI} + \frac{\partial u_i}{\partial y} \]

\[ \frac{\partial v}{\partial x} = \frac{Vp}{2EI} + \frac{\partial v_i}{\partial x} \]

SUBSTITUTING INTO THE EQUATION FOR SHEAR STRESS

\[ \frac{\partial u_i}{\partial y} - \frac{Px^2}{2EI} + \frac{Vp}{2EI} y^2 + \frac{\partial v_i}{\partial x} = -\frac{(1+\nu)p}{EI} (x^2 - y^2) \]

\[ \frac{\partial u_i}{\partial y} - \frac{Px^2}{2EI} + \frac{Vp}{2EI} y^2 + \frac{\partial v_i}{\partial x} = -\frac{(1+\nu)p}{EI} x^2 + \frac{(1+\nu)p}{EI} y^2 \]

\[ \frac{\partial u_i}{\partial y} + \left( \frac{Vp}{2EI} - \frac{(1+\nu)p}{EI} \right) y^2 = -\frac{\partial v_i}{\partial x} + \frac{p}{2EI} x^2 - \frac{(1+\nu)p}{EI} x^2 \]

\[ \frac{\partial u_i}{\partial y} - \frac{(2+\nu)p}{2EI} y^2 = -\frac{3V_i(x)}{\partial x} + \frac{p}{2EI} x^2 - \frac{(1+\nu)p}{EI} x^2 \]
Since left side of equation is dependent only on \( y \) and right side is dependent only on \( x \), each side must equal a constant.

\[
\frac{d^2 y}{dy^2} - \frac{(2 + v)P}{2EI} y^2 = -a_1
\]

\[
\frac{d^2 y}{dx^2} - \frac{P}{2EI} x^2 + \frac{(1 + v)P_h^2}{EI} = -a_1
\]

Integrating each equation

\[
y, (y) = \frac{(2 + v)P}{6EI} y^3 + a_1 y + a_2
\]

\[
y, (x) = \frac{P}{6EI} x^3 - \frac{(1 + v)P_h^2}{EI} x - a_1 x + a_3
\]

Substituting these into the displacement equations

\[
u = -\frac{P x^2 y}{2EI} + \frac{(2 + v)P}{6EI} y^3 + a_1 y + a_2
\]

\[
v = \frac{P x y^2}{2EI} + \frac{P}{6EI} x^3 - \frac{(1 + v)P_h^2}{EI} x - a_1 x + a_3
\]

Determine \( a_1, a_2, a_3 \) based on boundary conditions

At the fixed end: \( x = 0 \); \( y = 0 \) (i.e. N.A.)

\[
\frac{\partial u}{\partial y} = 0 \quad v = u = 0
\]
\[
\frac{du}{dy} = -\frac{P x^2}{2EI} + \frac{2(2+v)P}{2EI} y^2 + q_1
\]

@ \( x=L \), \( y=0 \)

\[
\frac{du}{dy} = -\frac{PL^2}{2EI} + a_1 = 0
\]

\[
a_1 = \frac{PL^2}{2EI}
\]

@ \( x=L \), \( y=0 \) \( u=0 \) \( y=0 \)

\[
u(0) = a_2 = 0
\]

\[
\nu(0) = \frac{PL^3}{6EI} - \frac{(1+v)Ph^2}{EI} L - \frac{PL^2}{2EI} (L) + q_3
\]

\[
a_3 = \frac{(1+v)Ph^2}{EI} L + \frac{PL^3}{3EI}
\]

**Now**

\[
u = -\frac{P x^2 y}{2EI} + \frac{(2+v)P}{6EI} y^3 + \frac{PL^2}{2EI} y + 0
\]

\[
u = \frac{(2+v)P}{6EI} y^3 + \frac{(L^2-x^2)P}{2EI} y
\]

\[
\nu = \frac{P x^3 y^2}{2EI} + \frac{P}{6EI} x^3 - \frac{(1+v)Ph^2}{EI} x - \frac{PL^2}{2EI} x + \frac{(1+v)Ph^2}{EI} L + \frac{PL^3}{3EI}
\]

\[
u = \frac{P}{EI} [\frac{x^3}{6} + \frac{L^3}{3} + \frac{x}{2} (\frac{1}{2}y^2-L^2) + \frac{1}{2} (1+v) L^2 (L-x^2)]
\]
FOR THE MOVEMENT OF THE N.A. \( y=0 \)

\( U = 0 \)  \( \text{(no horizontal movement due to small deformation assumption)} \)

\[
V = \frac{P x^3}{6 E I} - \frac{P L^2 x}{2 E I} + \frac{P L^3}{3 E I} + \frac{P h^2(1+\nu)}{E I} (L-x)
\]

\[
\frac{\partial V}{\partial x} = \frac{P x^2}{2 E I} - \frac{P L^2}{2 E I} - \frac{P h^2(1+\nu)}{E I}
\]

\[
\frac{\partial^2 V}{\partial x^2} = \frac{P x}{E I}
\]

RECALL EQ 5.7 OF TEXT

\[
\frac{1}{E I} \frac{\partial^2 V}{\partial x^2} = \frac{P x}{E I} = \frac{M}{E I}
\]

THIS IS THE SAME EQUATION OBTAINED FROM BEAM WITH COUPLE MOMENTS WE DERIVED FIRST.

FOR \( x=0 \)

\( V(0) = \frac{P L^3}{3 E I} + \frac{P h^2(1+\nu)L}{E I} \)

FROM ELEMENTARY BEAM BENDING

\[
V = \frac{P L^3}{3 E I} + \frac{P h^2 L}{2 G I}
\]

\[
\frac{P h^2 L}{2 G I} = \frac{3 P L}{2 G A} \Rightarrow \text{SHEAR DEFORMATION}
\]
Therefore, ratio of bending to shear deflection is a measure of slenderness.

\[
\frac{PA^2L/2GJ}{P^2L^3/3EI} = \frac{3}{2} \frac{4^2E}{L^2G} = \frac{3}{4} \frac{(1+\nu)(\frac{2L}{L})^2}{\frac{L}{L}} = \frac{3}{2} \frac{L}{L}
\]