CURVED BEAMS:

- Structural element with centroids of cross sections form a curve
- Assume centroids lie in single plane and cross section is narrow

First consider the basic relations in polar coordinates.

Polar coordinates are related to Cartesian coordinates as:

\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    r^2 &= x^2 + y^2 \\
    \theta &= \tan^{-1} \frac{y}{x}
\end{align*}
\]
\[
\frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta
\]
\[
\frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta
\]
\[
\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin \theta}{r}
\]
\[
\frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos \theta}{r}
\]

Now derivatives in \( x,y \) can be transformed into derivatives in \( r,\theta \) by using the chain rule:

\[
\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}
\]
\[
\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}
\]

For equilibrium.
Consider infinitesimal element abcd.

Body forces are \( F_r \neq F_\theta \)

\[
\begin{align*}
\varepsilon F_r &= 0 = \left( \frac{\partial F_r}{\partial r} dr \right) (r + dr) d\theta - \\
&- \frac{r}{r} r d\theta - \\
&- (\frac{\partial r}{\partial \theta} + \frac{\partial \theta}{\partial \theta}) dr \sin \frac{d\theta}{2} - \\
&- \frac{\partial \theta}{\partial \theta} \sin \frac{d\theta}{2} + \\
&+ (\frac{\partial r}{\partial \theta} + \frac{\partial \theta}{\partial \theta}) \cos \frac{d\theta}{2} - \\
&+ \frac{1}{2} r \cos \frac{d\theta}{2} + \\
&+ F_r r dr d\theta = 0
\end{align*}
\]

Now for small \( d\theta \)

\[
\sin \frac{d\theta}{2} = \frac{d\theta}{2},
\]

\[
\cos \frac{d\theta}{2} = 1,
\]

\[
\begin{align*}
\varepsilon F_r &= 0 = \frac{\partial}{\partial r} r d\theta + \left( \frac{\partial \theta}{\partial \theta} + \frac{\partial}{\partial r} \right) dr d\theta + \frac{\partial r}{\partial r} r^2 d\theta - \\
&- \frac{r}{r} r d\theta - \\
&- \frac{\partial \theta}{\partial \theta} \sin \frac{d\theta}{2} - \frac{\partial \theta}{\partial \theta} \sin \frac{d\theta}{2} - \\
&+ (\frac{\partial r}{\partial \theta} + \frac{\partial \theta}{\partial \theta}) \cos \frac{d\theta}{2} - \\
&+ \frac{1}{2} r \cos \frac{d\theta}{2} + \\
&+ F_r r dr d\theta = 0
\end{align*}
\]

\[
\begin{align*}
\varepsilon F_r &= \frac{\partial F_r}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{\partial F_r}{\partial \theta} + F_r = 0
\end{align*}
\]

\[
\begin{align*}
\varepsilon F_\theta &= \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial r}{\partial \theta} + \frac{2}{r} \frac{\partial F_\theta}{\partial \theta} + F_\theta = 0
\end{align*}
\]
IF $r \neq 0$ THE EQUATIONS OF EQUILIBRIUM ARE SATISFIED BY THE STRESS FUNCTION $\phi(r, \theta)$

$$\sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\tau = \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} - \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) = - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right).$$

NOW CONSIDER THE DEFORMATION OF INFINITESIMAL ELEMENT $abcd$

DISPLACEMENT IN $r$ IS $u$

DISPLACEMENT IN $\theta$ IS $v$

DEFORMATION IS RESULT OF CHANGE IN LENGTH AND ROTATION
Now if \( \sin \theta \neq 0 \)

ASSUME LINES AD & CD ARE ESSENTIALLY STRAIGHT (SMALL ANGLE & SMALL DEFORMATIONS)

NOW A RADIAL DISPLACEMENT OF SIDE AB
RESULTS IN BOTH RADIAL & TANGENTIAL STRAINS

RADIAL STRAIN \( \varepsilon \) ASSOCIATED WITH
U DISPLACEMENT OF AD IS

\[
\varepsilon_r = \frac{\partial u}{\partial r}
\]

THE TANGENTIAL STRAIN ASSOCIATED WITH
U DISPLACEMENT IS

\[
(\varepsilon_\theta)_u = \frac{(r+u) d\theta - rd\theta}{r d\theta} = \frac{u}{r}
\]

ALSO, A \( \theta \)-DISPLACEMENT PRODUCES
TANGENTIAL STRAIN

\[
(\varepsilon_\theta)_\theta = \frac{1}{r} \frac{\partial \theta}{\partial \theta} = \frac{1}{r} \frac{\partial \theta}{\partial \theta}
\]

THEREFORE THE TANGENTIAL STRAIN IS

\[
\varepsilon_\theta = \frac{1}{r} \frac{\partial \theta}{\partial \theta} + \frac{u}{r}
\]
Now the angle of rotation \( \theta_f \) of side \( ab' \) due to a \( u \) displacement produces shear strain.

\[
(\gamma_{r\theta})_u = \frac{(\partial u/\partial \theta) d\theta}{r d\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta}
\]

The rotation of side \( bc \) associated with a \( v \) displacement is shown in (d) as \( gb''h \). The shear strain is:

\[
(\sigma_{r\theta})_v = \frac{\partial v}{\partial r} - \frac{v}{r}
\]

Therefore the total shear strain is:

\[
\sigma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}
\]

So the strain-displacement relationships for polar coordinates are:

\[
\varepsilon_r = \frac{\partial u}{\partial r}
\]

\[
\varepsilon_\theta = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}
\]

\[
\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}
\]
Now Hooke's Law in polar coordinates is:

\[ \epsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \]
\[ \epsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \]
\[ \gamma_{r\theta} = \frac{1}{G} \tau_{r\theta} \]

Specialized for plane strain

\[ \epsilon_r = \frac{1 + \nu}{E} [(1-\nu) \sigma_r - \nu \sigma_\theta] \]
\[ \epsilon_\theta = \frac{1 + \nu}{E} [(1-\nu) \sigma_\theta - \nu \sigma_r] \]
\[ \gamma_{r\theta} = \frac{1}{G} \tau_{r\theta} \]

Transformation equations are arrived at by replacement of variables

\[ \sigma_r = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \]
\[ \sigma_\theta = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2 \tau_{xy} \sin \theta \cos \theta \]
\[ \tau_{r\theta} = (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \]
Finally, compatibility equations from the strain-displacement equations

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}
\]

\[
\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial \varepsilon_\theta}{\partial r} + \frac{\mu}{r} \frac{\partial \gamma_{rr}}{\partial r} - \frac{1}{r} \frac{\partial \gamma_{\theta r}}{\partial \theta} = \frac{1}{r} \frac{\partial \gamma_{\theta \theta}}{\partial r} + \frac{1}{r^2} \frac{\partial \gamma_{\theta \theta}}{\partial \theta}
\]

To change to terms of the stress function \( \Phi \), the partial derivatives \( \frac{\partial^2 \Phi}{\partial x^2} \) and \( \frac{\partial^2 \Phi}{\partial y^2} \) must be evaluated.

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}
\]

This is the Laplacian operator.

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}
\]

An alternative form of the compatibility equation for axisymmetric forces and zero body forces.

\[
\nabla^2 (\sigma_r + \sigma_\theta) = \frac{\partial^2 (\sigma_r + \sigma_\theta)}{\partial r^2} + \frac{1}{r} \frac{\partial (\sigma_r + \sigma_\theta)}{\partial r} = 0
\]
Now back to curved beams:

- Apply constant moment.
- Stress distribution is independent of $\theta$.

Now for equilibrium:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

For compatibility in plane stress:

$$\frac{d^2 (\sigma_r + \sigma_\theta)}{dr^2} + \frac{1}{r} \frac{d}{dr} \left( \frac{d (\sigma_r + \sigma_\theta)}{dr} \right) = 0$$

If $r = e^t$ or $t = \ln r$ is substituted and integrated, you have:

$$\sigma_r + \sigma_\theta = c'' + c' \ln r$$

This can be rewritten as:

$$\sigma_r + \sigma_\theta = c'' + c' \ln \left( \frac{r}{a} \right)$$
SOLVING ALONG WITH THE EQUILIBRIUM EQUATION

\[ \sigma_r = C_1 + C_2 \ln \frac{r}{a} + \frac{C_3}{r^2} \]

\[ \sigma_\theta = C_1 + C_2 (1 + \ln \frac{r}{a}) - \frac{C_3}{r^2} \]

**USE BOUNDARY CONDITIONS TO SOLVE FOR CONSTANTS**

\( \sigma \) at \( r = a \) or \( r = b \) \( \sigma_r = 0 \)

\( \theta \) ENDS THERE ARE NO NORMAL FORCES

\[ \sigma \int_a^b \theta \ d\tau = 0 \]

\( \theta \) ENDS \( \int_a^b \theta \ d\tau = M \)

\[ \sigma \int_a^b \theta \ d\tau = M \]

**ALSO** \( \theta |_{\theta = 0}^\theta = 0 \)

**THIS RESULTS IN**

\[ C_3 = -a^2 C_1 \]

\[ C_1 (\frac{a^2}{b^2} - 1) = C_2 \ln \frac{b}{a} \]

AND WITH M B.C.

\[ C_1 = \frac{b^2 \ln (b/a)}{a^2 - b^2} C_2 \]

\[ C_3 = \frac{a^2 b^2 \ln (b/a)}{b^2 - a^2} C_2 \]
Finally substituting into (6)

\[ c_2 = \frac{M}{N} \frac{4(6^2 - a^2)}{c b^4} \]

\[ N = \left(1 - \frac{a^2}{b^2}\right)^2 - 4 \frac{a^2}{b^2} \ln \frac{2 b}{a} \]

Substituting

\[ b_1^* = \frac{4 M}{c b^2 N} \left[ \left(1 - \frac{a^2}{b^2}\right) \ln \frac{r}{a} - \left(1 - \frac{a^2}{r^2}\right) \ln \frac{b}{a} \right] \]

\[ b_2^* = \frac{4 M}{c b^2 N} \left[ \left(1 - \frac{a^2}{b^2}\right) \left(1 + \ln \frac{r}{a}\right) - \left(1 + \frac{a^2}{r^2}\right) \ln \frac{b}{a} \right] \]