CRITERIA FOR MATERIAL FAILURE

YIELDING:

INITIATE AS LOCAL STRAINS AND IN DUCTILE MATERIALS PROGRESSES TO GENERAL YIELDING.

DUCTILE VS BRITTLE

CREEP - DEFORMATION OVER TIME WHEN SUBJECTED TO A CONSTANT STRESS

RELAXATION - CHANGE IN STRESS OVER TIME WHEN SUBJECTED TO A CONSTANT STRAIN.
Consider:

\[ \sigma = \frac{F}{A} = \sigma_{yp} \quad (\text{or } \varepsilon) \]

2. \[ \varepsilon = \frac{1}{2} (\varepsilon_p) \quad \Rightarrow \quad \varepsilon_p = \frac{1}{2} \sigma_{yp} \quad \varepsilon = (\varepsilon_p - \varepsilon_0) \]

3. \[ \varepsilon = \varepsilon_{yp} \quad (\text{or } \varepsilon_0) \]

4. Total Strain Energy
   \[ U_{yp} = \frac{1}{2} (\sigma_{yp}^2 / \varepsilon) \]

5. Distortional Energy
   \[ U_{de} = \frac{1}{2} (1 + 2\mu) / 3E \int \varepsilon_0^2 \]
6. \( \varepsilon_{oct} = \left(\frac{V^2}{3}\right)_{yp} \)

WHAT IS THE RELATIONSHIP BETWEEN \( \varepsilon_{oct} \) AND \( \varepsilon_{yf} \)?

WE KNOW THAT

\[
\varepsilon_{oct} = \frac{1}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}
\]

NOW MULTIPLY BY \( \frac{\sqrt{6E}}{6\varepsilon} \sqrt{\frac{1+\nu}{1+\nu}} \)

\[
\therefore \varepsilon_{oct} = \frac{1}{3} \sqrt{\frac{6E}{1+\nu}} \left[ (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right] \frac{\sqrt{1+\nu}}{6\varepsilon}
\]

HOWEVER:

\[
\varepsilon_{yf} = \frac{1+\nu}{3E} \left[ (\varepsilon_1 - \varepsilon_2)^2 + \ldots \right]
\]

\[
= \frac{1+\nu}{6\varepsilon} \left[ (\varepsilon_1 - \varepsilon_2)^2 + \ldots \right]
\]

\[\therefore \varepsilon_{oct} = \frac{1}{3} \sqrt{\frac{6E}{1+\nu}} \varepsilon_{yf} \]

HOWEVER:

\[
\varepsilon_{yf} = \frac{1+\nu}{3E} \varepsilon^2 \quad \text{(ER C P 85)}
\]

\[\therefore \text{AT FAILURE (i.e., YIELD)}
\]

\[
\varepsilon_{oct} = \frac{1}{3} \sqrt{\frac{6E}{1+\nu}} \frac{(1+\nu)}{3E} \varepsilon^2
\]
\[ \varepsilon_{\text{oct}} = \frac{\sqrt{2}}{3} \varepsilon_3 = 0.47 \varepsilon_3. \]

Consider 3D:

\[ \sigma_1 \geq \sigma_2 \geq \sigma_3 \]

Now uniaxial results in

\[ \sigma_1 = \frac{\sigma_{up}}{2} \]

\[ \sigma_2 = \sigma_3 = 0 \]

\[ \varepsilon_{up} = \frac{\sigma_{up}}{E} \]

**Maximum Distortional Strain Energy**

\[ U_{dd} = \frac{(1 + v)}{3E} \int \sigma_{up}^2 \]

**Maximum Octahedral Shear Stress**

\[ \varepsilon_{\text{oct}} = \left( \frac{\sqrt{2}}{3} \right) \varepsilon_{up}. \]
Maximum Shearing Stress Theory

(Tresca Yield Criterion)

Yielding occurs when maximum shear stress equals maximum shear stress in a simple tension test.

\[ \frac{1}{2} | \sigma_1 - \sigma_3 | = \tau_{yr} = \frac{1}{2} \tau_{tp} \]

\[ | \sigma_1 - \sigma_3 | = \tau_{yr} \]

For plane stress, \( \sigma_3 = 0 \).

\[ | \sigma_1 - \sigma_2 | = \tau_{yr} \]

Consider when \( \sigma_1 \neq \sigma_2 \) are tension or compression.

\[ \frac{\sigma_1 - \sigma_2}{\tau_{yr}} = \frac{\sigma_2}{\tau_{yr}} \]

\[ | \sigma_1 - \sigma_2 | = \tau_{yr} \]

Or:

\[ \frac{\sigma_1}{\tau_{yr}} - \frac{\sigma_2}{\tau_{yr}} = \pm 1 \]

For both tension or both compression.

\[ \frac{\sigma_1 - \sigma_3}{2} = \frac{\tau_{yr}}{2} \]

For \( |\sigma_1| > |\sigma_2| \):

\[ |\sigma_1| = \tau_{yr} \]

\[ |\sigma_2| > |\sigma_1| \]

\[ |\sigma_2| = \tau_{yr} \]
MAXIMUM DISTORTION ENERGY THEORY (i.e. von Mises Theory)

Yielding occurs when the distortional energy at any point equals the distortional energy in a simple tension test.

\[ U_{xx} = \frac{3}{2} \frac{E}{G} \left( \frac{1 + \nu}{3(1 - 2\nu)} \right) \sigma_y^2 \]

\[ U_{xx} = \frac{3}{2} \frac{E}{G} \left[ \frac{1}{\nu} \left( (\varepsilon_x - \sigma_y) + (\varepsilon_y - \sigma_y) + (\varepsilon_z - \sigma_y) \right)^2 + 6 (\varepsilon_{xy})^2 + (\varepsilon_{xz})^2 + (\varepsilon_{yz})^2 \right] \]

\[ = \frac{1 - \nu}{3(1 - 2\nu)} \sigma_y^2 \]
\[ \sigma = \sqrt{\frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]} \]

The Mises stress equation is used to determine the equivalent stress in a material under complex stress states. The equation captures the maximum shear stress acting on a material, allowing engineers to assess the material's capacity to withstand stress without damage. The stress components \( \sigma_1, \sigma_2, \sigma_3 \) correspond to the principal stresses in a given direction, which are critical in determining the material's behavior under load.
OCTAHEDERAL SHEARING STRESS THEORY.
(MISES-HENCKY OR VON MISES CRITERION)

\[ \tau_{oct} = \text{CONSTANT:} \]

Usually the constant is some percentage of \( \sigma_p \).

\[ \text{i.e., } \quad 0.47 \sigma_p \]

\[ \tau_{oct} = \frac{3}{2} \sqrt{\left( \sigma_x - \sigma_y \right)^2 + \left( \sigma_y - \sigma_z \right)^2 + \left( \sigma_z - \sigma_x \right)^2 + 6 \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2} \]

In terms of distortional energy

\[ U_{OD} = \frac{3}{2} \left( \frac{1+v}{E} \right) \tau_{oct}^2 \]

Essentially, criterion is a shorthand method of dealing with energy but measuring stress only.
σ_y = 350 MPa

M = 8 kN.m  T = 24 kN.m

Find d for F.O.S. of 2.0

σ_x is due to moment

σ_y is due to torque

\[ \sigma_x = \frac{M_y}{I_y} = \frac{32 M}{\Pi d^3} \]

\[ \tau_{xy} = \frac{T}{J} = \frac{16 T}{\Pi d^3} \]

σ_y = 0  τ_z = 0

\[ \sigma_{max, min} = \frac{\sigma_x^2 + \sigma_y^2}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

\[ \varepsilon_{max} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

\[ \varepsilon_{max, min} = \frac{16 M}{\Pi d^3} + \sqrt{\left(\frac{16 M}{\Pi d^3}\right)^2 + \left(\frac{16 T}{\Pi d^3}\right)^2} \]

\[ \varepsilon_{max} = \pm \sqrt{\left(\frac{16 M}{\Pi d^3}\right)^2 + \left(\frac{16 T}{\Pi d^3}\right)^2} \]
\[
\sigma_{\text{max}} = \frac{16(8)}{\pi d^3} + \sqrt{\left(\frac{16(8)^2}{\pi d^3}\right) + \left(\frac{16(24)^2}{\pi d^3}\right)}
\]

\[
\sigma_{\text{max}} = \frac{40.74}{d^3} + \sqrt{\frac{1660}{d^9} + \frac{14940}{d^9}}
\]

\[
\sigma_{\text{max}} = \frac{40.74}{d^3} + \frac{128.8}{d^3}
\]

Because 2nd term is larger than 1st condition is than \( \sigma \) is tension and \( \sigma \) is compression.

.. Eq 4.20 governs

\[
16 - 0.51 = \sigma_p
\]

For problem

\[
16 - 0.51 = \frac{\sigma_p}{\text{F.o.s.}}
\]

\[
\sqrt{\frac{x^2 + 4y^2}{xy}} = \frac{\sigma_p}{\text{F.o.s.}}
\]

\[
\sqrt{\left(\frac{32y^2}{\pi d^3}\right) + \left(\frac{16T}{\pi d^3}\right)^2} = \frac{\sigma_p}{\text{F.o.s.}}
\]

\[
\sqrt{\left(\frac{32(8)^2}{\pi d^3}\right) + \left(\frac{16(24)^2}{\pi d^3}\right)^2} = \frac{350}{2.0}
\]

\[d = 113.8 \text{ mm}\]
Example 4.2

A steel conical tank, supported at its edges, is filled with a liquid of density $\gamma$ (Fig. P13.17). The yield point stress ($\sigma_{yp}$) of the material is known. The cone angle is $2\alpha$. Determine the required wall thickness $t$ of the tank based on a factor of safety $f_s$. Apply (a) the maximum shear stress theory and (b) the maximum energy of distortion theory.
Sec. 4.7  Octahedral Shearing Stress Theory

Solution  The variations of the circumferential and longitudinal stresses in the tank are, respectively (Prob. 13.17),

\[ \sigma_1 = \gamma (a - y) y \frac{\tan \alpha}{t \cos \alpha}, \quad \sigma_2 = \gamma (a - \frac{2}{3} y) y \frac{\tan \alpha}{2t \cos \alpha} \]  \hspace{1cm} (e)

The principal stresses have the largest magnitude:

\[ \sigma_{1,\text{max}} = \frac{\gamma a^2 \tan \alpha}{4t \cos \alpha}, \quad \text{at} \quad y = \frac{a}{2} \]  \hspace{1cm} (d)

\[ \sigma_{2,\text{max}} = \frac{3\gamma a^2 \tan \alpha}{16t \cos \alpha}, \quad \text{at} \quad y = \frac{3a}{4} \]

(a) **Maximum shear stress theory:** Because \( \sigma_1 \) and \( \sigma_2 \) are of the same sign and \( |\sigma_1| > |\sigma_2| \), we have, from the first equation of (4.3) together with (d),

\[ \frac{\sigma_{\text{yp}}}{f_s} = \frac{\gamma a^2 \tan \alpha}{4t \cos \alpha} \]

The thickness of the tank is found from this equation to be

\[ t = 0.250 \frac{\gamma a^2 f_s \tan \alpha}{\sigma_{\text{yp}} \cos \alpha} \]  \hspace{1cm} (e)

(b) **Maximum distortion energy theory:** It is observed in Eq. (d) that the largest values of principal stress are found at different locations. We shall therefore first locate the section at which the combined principal stresses are at a critical value. For this purpose, we insert Eq. (e) into Eq. (4.5a):

\[ \frac{\sigma_{\text{yp}}^2}{f_s^2} = \left[ \gamma (a - y) y \frac{\tan \alpha}{t \cos \alpha} \right]^2 + \left[ (a - \frac{2}{3} y) y \frac{\tan \alpha}{2t \cos \alpha} \right]^2 \]

\[ - \left[ \gamma (a - y) y \frac{\tan \alpha}{t \cos \alpha} \right] \left[ (a - \frac{2}{3} y) y \frac{\tan \alpha}{2t \cos \alpha} \right] \]  \hspace{1cm} (f)

Upon differentiating Eq. (f) with respect to the variable \( y \) and equating the result to zero, we obtain

\[ y = 0.52a \]

Upon substitution of this value of \( y \) into Eq. (f), the thickness of the tank is determined:

\[ t = 0.225 \frac{\gamma a^2 f_s \tan \alpha}{\sigma_{\text{yp}} \cos \alpha} \]  \hspace{1cm} (g)

The thickness based on the maximum shear stress theory is thus 10% larger than that based on the maximum energy of distortion theory.
COMPARE THEORIES.

MAX SHEAR STRESS

\( \tau_p = 0.50 \text{ bar} \)

MAX DISTORTIONAL ENERGY

\( \tau_p = 0.577 \text{ bar} \)
Maximum shear stress

\[ \sigma_{yp} = \sqrt{\sigma_x^2 + 4 \tau_{xy}^2} \]

\[ \sigma_x = \frac{4P}{\pi d^2} \]

\[ \tau_{xy} = \frac{16T}{\pi d^3} \]

Maximum distortion energy

\[ \sigma_{yp} = \sqrt{\sigma_x^2 + 3 \tau_{xy}^2} \]
MAXIMUM PRINCIPAL STRESS THEORY
(BRITTLE MATERIALS)

\[ \sigma_1 = \sigma_y \text{ ult.} \]

\[ |\sigma_1| = \sigma_y \quad \text{or} \quad |\sigma_3| = \sigma_y \]

\[ \sigma_2/\sigma_y \]

\[ \sigma_3/\sigma_y \]

\[ (-1,1) \quad (1,1) \quad (1,-1) \quad (-1,-1) \]

\[ (0,0) \]
MOHRI'S THEORY
(REAL BRITTLE MATERIALS)

(a)

Simple compression

(b)

Torsion

Failure envelope

Simple tension
Coulomb-Mohr Theory (Friction)

\[ \tau = \sigma \alpha + b \]

\( \alpha \) and \( b \) are material properties

![Diagram of failure envelope and simple compression and tension](image)

\[ \tau = \frac{\sigma_1 - \sigma_2}{2} \quad \text{and} \quad \sigma = \frac{\sigma_1 + \sigma_2}{2} \]

Now, substituting

\[ \frac{\sigma_1 - \sigma_2}{2} = \alpha \left( \frac{\sigma_1 + \sigma_2}{2} \right) + b \]

\[ \sigma_1 - \sigma_2 = 2\alpha \sigma_1 + \alpha \sigma_2 + 2b \]

\[ (1-a) \sigma_1 - (1+a) \sigma_2 = 2b \]

Under simple tension

\[ \sigma_1 = \sigma_u \quad \text{and} \quad \sigma_2 = 0 \]

Under simple compression

\[ \sigma_2 = \sigma_u \quad \text{and} \quad \sigma_1 = 0 \]
\[ \sigma_y (1 - q) = 2b \quad \sigma_y' (1 + q) = 2b \]

\[ a = \frac{\sigma_y - \sigma_y'}{\sigma_y + \sigma_y'} \quad b = \frac{\sigma_y \sigma_y'}{\sigma_y + \sigma_y'} \]

Substituting

\[ (1 - \frac{\sigma_y - \sigma_y'}{\sigma_y + \sigma_y'}) \sigma = (1 + \frac{\sigma_y - \sigma_y'}{\sigma_y + \sigma_y'}) \sigma_2 = 2 \frac{\sigma_y \sigma_y'}{\sigma_y + \sigma_y'} \]

If \( \sigma_1 \) is TENSION and \( \sigma_2 \) is COMPRESSION, then

\[ \frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1} = 1 \] is FAILURE

For both COMPRESSION or TENSION

MOHR's CIRCLE DEFINES FAILURE SURFACE

Simple compression

Simple tension

Failure envelope

For PURE SHEAR,

\[ \sigma_1 = -\sigma_2 \]

\[ \therefore \sigma_1 = \sigma_2 = \frac{\sigma_y \sigma_y'}{\sigma_y + \sigma_y'} \]
METAL FATIGUE

$$\sigma_m = \frac{1}{2} (\sigma_{\text{max}} - \sigma_{\text{min}})$$
$$\sigma_a = \frac{1}{2} (\sigma_{\text{max}} - \sigma_{\text{min}})$$

**FATIGUE IS A LOCALIZED PHENOMENON THAT PROPAGATES.**

**FOR FULL-REVERSAL, \( \sigma_m \) IS LESS IMPORTANT AND \( \sigma_a \) DRIVES ANALYSIS**

\( \sigma_{\text{max}} \) IS INITIATOR FOR NON-REVERSAL LOADING

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**TABLE 4.2**

<table>
<thead>
<tr>
<th>Fatigue criterion</th>
<th>Modified Goodman</th>
<th>Soderberg</th>
<th>Gerber</th>
<th>SAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation</td>
<td>( \frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_u} = 1 )</td>
<td>( \frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_{yp}} = 1 )</td>
<td>( \frac{\sigma_a}{\sigma_{cr}} + \left( \frac{\sigma_m}{\sigma_u} \right)^2 = 1 )</td>
<td>( \frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_f} = 1 )</td>
</tr>
</tbody>
</table>

*a Subscript notations: \( a = \) alternating, \( yp = \) static tensile yield, \( m = \) mean, \( u = \) static tensile ultimate, \( cr = \) completely reversed, \( f = \) fracture.*

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**MEAN STRESS-ALTERNATING STRESS RELATIONS**
Fatigue strength will have a value between fracture and endurance stress for a given number of cycles.

![Graph of Stress (Pa) vs Fatigue life (cycles)](image)

**Figure 4.3** Typical S–N diagram for steel.

Experience has shown:

1) Steel performance is predicted well by Soderberg or modified Goodman

2) Hardened steels follow SAE and modified Goodman since they have identical solutions for brittle materials $\sigma_b = \sigma$
Figure 16.1 $\sigma - N$ diagram.

Figure 16.2 $\sigma - N$ diagrams for three metals.
Figure 16.3 $\sigma - N$ band indicating scatter of fatigue data.
FATIGUE UNDER COMBINED LOADING.

COMBINED STRESS CONDITIONS CAN ACCENTUATE FATIGUE PROBLEMS.

A METHOD OF ADDRESSING IS AS FOLLOWS

\[ N_{cr} = N_f \left( \frac{\sigma_{cr}}{\sigma_f} \right)^{\frac{1}{b}} \]

WHERE

\[ b = \frac{\ln(\sigma_f/\sigma_0)}{\ln(N_f/N_0)} \]

\( \sigma_0 \) & \( \sigma_f \) ARE DEFINED IN TERMS OF SIMPLE TENSION STRESSES.
### TABLE 4.3

End points for $S$–$N$ diagram (Fig. 4.3)

<table>
<thead>
<tr>
<th>Fatigue Criterion</th>
<th>Ductile steels ($\sigma_u \leq 1750$ MPa)</th>
<th>Brittle (hard) steels ($\sigma_u &gt; 1750$ MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_i$</td>
<td>$N_f$</td>
</tr>
<tr>
<td>Modified</td>
<td>0.9$\sigma_u$</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Goodman</td>
<td>0.9$\sigma_u$</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Soderberg</td>
<td>0.9$\sigma_u$</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Gerber</td>
<td>$\sigma_u + 350 \times 10^6$</td>
<td>1</td>
</tr>
</tbody>
</table>

*state of stress. According to the method of life, $N_e$ (Fig. 4.3) is 1.8 $\times 10^8$.

![Figure 4.3 Typical S–N diagram for steel.](image)
IMPACT OR DYNAMIC LOADS.

ASSUMPTIONS:

1. A \& F (STATIC OR DYNAMIC)
2. INERTIAL EFFECTS CAN BE NEGLECTED
3. ELASTIC RESPONSE
4. ENERGY IS CONSERVED

\[ W \]

\[ h \]

\[ k \]

**Figure 4.13** Freely falling body.

\[ \text{START } \quad V_0 = 0 \]

\[ \text{MAX } A \quad V_f = 0 \]

\[ W (h + \delta_{\text{max}}) - \frac{1}{2} k \delta_{\text{max}}^2 = 0 \]

\[ \Delta \text{ POTENTIAL OF WEIGHT} \quad \uparrow \]

\[ \text{POTENTIAL OF SPRING} \quad \uparrow \]
$W/\kappa = \delta_{\text{static}}$

$\delta_{\text{max}} = \delta_{\text{static}} + \sqrt{\left(\delta_{\text{static}}\right)^2 + 2 \delta_{\text{static}} h}$

REARRANGE

$\delta_{\text{max}} = \delta_{\text{static}} \left(1 + \left(1 + \frac{2h}{\delta_{\text{static}}}ight)\right)$

THE IMPACT FACTOR.

$\delta_{\text{max}} = \frac{1}{\delta_{\text{static}}} \sqrt{1 + \frac{2h}{\delta_{\text{static}}}}$

$P_{\text{yn}} = W \left(1 + \sqrt{1 + \frac{2h}{\delta_{\text{static}}}}\right)$

WHEN $h >> \delta_{\text{static}}$

$W/\delta_{\text{max}}$ is NEGREGABLE TERM.

WHEN $h = 0$, $\delta_{\text{max}} = 2 \delta_{\text{static}}$.

FOR HORIZONTAL MOTION KINETIC ENERGY replaces WEIGHT POTENTIAL ENERGY AND.

$Wv^2/2g + \frac{1}{2} k S_{\text{max}}^2 = 0$

$P_{\text{yn}} = W \frac{v^2}{2g \delta_{\text{static}}}$, $\delta_{\text{max}} = \delta_{\text{static}} \sqrt{\frac{v^2}{g \delta_{\text{static}}}}$