Shear center is sometimes called the flexural center.

It is the point in the cross section where the resultant of the shear stresses pass.

Non-symmetric and open sections cause the center to lie outside the cross section at times.

Use EMA about an arbitrary point A to determine the position.

\[ \Sigma M_x = SS (\tau_{xy} z - \tau_{xz} y) \, dz \, dy \]

Now \( e_z = \frac{M_x}{V_z} \)

\( e_y = \frac{M_x}{V_y} \)

Some as finding centroid using moments of area.

Note: Vector of force to cause bending only may not be oriented along initial axis. It is oriente along principle axis.
EXAMPLE:

Locate the shear center for the cross section.

The centroid is located:

\[ \bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{(125)(12.5)(3.75)+(37.5)(125)(37.5)}{(125)(125)+(37.5)(125)+(25)(1.25)} \]

\[ \bar{y} = 15.63 \text{ mm} \]

\[ \bar{z} = \frac{(125)(12.5)(0)+(37.5)(125)(0)+(25)(1.25)(25)}{(125)(125)+(37.5)(125)+(25)(1.25)} \]

\[ \bar{z} = 5.21 \text{ mm} \]

\[ I_y = \sum (y_i^2 + A_i y_i^2) \]

\[ = \frac{(12.5)(125)^3}{12} + \frac{(12.5)(125)(37.5-15.63)^2}{2} + \frac{(37.5)(125)^3}{12} + \frac{(125)(37.5)(37.5-15.63)^2}{2} + \frac{(25)(1.25)^3}{12} + (25)(1.25)(15.63)^2 \]

\[ I_y = 4765.62 \text{ mm}^4 \]

\[ I_z = 29,057.69 \text{ mm}^4 \]
\[ I_{xy} = \sum (x^2 + Ady) dA = 3984.37 \text{ mm}^4 \]

**Now the principle axes are oriented at an angle} \theta. where**

\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = y \cos \theta + z \sin \theta \]

**Therefore:**

\[ I_{x'} = \sum y'^2 dA = \sum (y \cos \theta - 2 \sin \theta)^2 dA \]
\[ = \cos^2 \theta \sum y^2 dA + \sin^2 \theta \sum z^2 dA - 2 \sin \theta \cos \theta \sum xy dA \]

**Now using**

\[ I_y = \sum z^2 dA \]

**and**

\[ I_{yz} = \sum yz dA \]

\[ I_{x'} = I_x \cos^2 \theta + I_y \sin^2 \theta - 2 I_{xy} \sin \theta \cos \theta \]

\[ I_y \] is found by substituting \( \theta + \frac{\pi}{2} \) for \( \theta \)

**And**

\[ I_{z'} = \sum y'^2 dA \]

**So**

\[ I_{y'} = (I_y - I_z) \sin^2 \theta + I_{yz} (\cos^2 \theta - \sin^2 \theta) \]

**Now using double-angle trig identities**
\[ I_{y'} = \frac{I_y + I_z}{2} + \frac{I_y - I_z}{2} \cos \theta - I_{yz} \sin \theta \]

\[ I_z' = \frac{I_y + I_z}{2} - \frac{I_y - I_z}{2} \cos \theta + I_{yz} \sin \theta \]

\[ I_{y'}' = -\frac{I_y - I_z}{2} \sin \theta + I_{yz} \cos \theta \]

These are same for a for transform stress and strain.

\[ (I_y - I_z) \sin \theta + 2I_{yz} \cos \theta = 0 \]

And

\[ \tan \theta = -\frac{2I_{yz}}{I_y - I_z} \]

Therefore, \[ \theta = \frac{2(3984)}{(4965 - 2105)} = 13^\circ \]

And

\[ I_{y'} = \frac{4965 + 2105}{2} + \frac{4965 - 2105}{2} \cos(2(13)) \]

\[ - (3984) \sin(2(13)) = 3828 \text{ mm}^4 \]

\[ I_z = 21953 \text{ mm}^3 \]

Assume \( V \) is applied which is the resultant of the shear forces in legs of cross section.
To find magnitude of $F_1$ in flange:

$$
\varepsilon_{x_2} = \frac{V_y \cdot Qy'}{I_y \cdot b} = \frac{V_y}{I_y} \int_{-5}^{5} (19.55 + \frac{1}{2} 55 \sin 13.0^\circ) \, dx
$$

Substituting values and integrating across the flange:

$$
F_1 = \int_{0}^{5} \varepsilon_{x_2} \, cdS = \frac{V_y}{I_y} \int_{0}^{5} 12.5 \cdot \int_{-5}^{5} (19.55 + \frac{1}{2} 55 \sin 13.0^\circ) \, dx
$$

$$
F_1 = 0.0912 V_y
$$

Now $\Sigma \phi_y = 0$ results in:

$$
V_y \cdot e_2' = 37.5 F_1
$$

And then:

$$
e_2' = 3.42 \text{ mm}
$$

Again for $V_2$:

$$
\varepsilon_{x_2} = \frac{V_2 \cdot Qy'}{I_y \cdot b} = \frac{V_2}{I_y} \int_{-5}^{5} (12.05 + \frac{1}{2} 55 \cos 13.0^\circ) \, dx
$$
\[ F = \frac{V_2}{I_y} \int^x_{12.5} 52(12.05 - \frac{1}{2}5500(13^\circ)) \, dx \]

\[ = 0.204 V_2 \]

Now \( \sum M = 0 \)

\[ V_2 e_y' = 37.5 F_1 = 7.65 V_2 \]

AND

\[ e_y' = 7.65 \text{ mm} \]
**Shear Centers:**

\[ b = \frac{M y}{V} \]

**FREE EDGE**

\[ \sigma + \Delta \sigma \]

\[ \sigma \]

\[ \sigma \]

\[ b \]

\[ F + dF \]

\[ \int b dx \]

\[ F = \int b dx \]

\[ EF = 0 = F + 26 dx \quad (F + dF) \]

\[ 26 dx = dF = \int dF = \int \frac{dM}{dx} \frac{dy}{dx} dA \]

\[ \frac{dM}{dx} = V \]

\[ \therefore \quad \varepsilon = \frac{V}{6 E I} \int y dA \]

\[ \varepsilon \]

**Static Moment of Area from Cut of Interest to Edge (Shaded Area):**

\[ Z = \frac{V Q}{6 E I} \]
This method can be used with symmetric sections but cannot solve non-symmetric sections, i.e., unequal thickness or shape.

Magnitude depends on relative stiffness.

This is an indeterminate problem. Either use flexibility or stiffness methods.

Must have principle centroidal axis.
\[ \sigma = -\frac{My}{I} \]

F.B.D.

\[ S_0 d\sigma dA + q \, dx - q_0 \, dx = 0 \]

\[ d\sigma \frac{d\sigma}{dx} \, dx + q \, dx - q_0 \, dx = 0 \]

\[ S_0 \frac{d\sigma}{dx} dA + q \, - q_0 = 0 \]

\[ q = q_0 - S_0 \frac{d\sigma}{dx} dA \]
now: \( \frac{dV}{dx} = -\frac{V_y}{I} \)

\[
Q = Q_0 + \frac{V}{I} \int_0^s y dA
\]

\[
\frac{dy}{ds} = \frac{I}{g} - v \phi
\]

\[
\frac{du}{ds} = \frac{q}{gt} - v \phi
\]

since \( V @ shear \text{ center}, \phi = 0 \)

\[
\int du = \int \frac{q}{gt} ds
\]

\[
U - U_0 = \int_0^s \frac{q ds}{gt}
\]

now if the entire parameter is considered, there can not be a discontinuity

\[
U - U_0 = 0
\]

\[
0 = \int \frac{q ds}{gt}
\]

substitute boxed equation

\[
\int \frac{q ds}{gt} + \int \left[ \frac{V}{I} \int_0^s y dA \right] \frac{ds}{gt} = 0
\]

solve for \( Q_0 \)

\[
Q_0 \int \frac{ds}{gt} = -\frac{V}{I} \int \left[ \int_0^s y dA \right] \frac{ds}{gt}
\]
However:

\[ S uv = uv - S vdu \]

\[ u = \int_0^s y \, dA \quad dv = \frac{ds}{gt} \]

\[ du = y \, dA \quad v = \int_0^s \frac{ds}{gt} \]

Substituting,

\[ -\frac{V}{I} \left[ \left( \int y \, dA \right) \int_0^s \frac{ds}{gt} \right] - \int_0^s \frac{ds}{gt} y \, dA \left( \int y \, dA \right) = q_0 \frac{ds}{gt} \]

\[ \left( \int y \, dA \right) \int_0^s \frac{ds}{gt} = 0 \quad \text{Area moment about gravity axis} \]

\[ q_0 = \frac{V}{I} \left[ \int \left[ \frac{\int_0^s \frac{ds}{gt}}{\int y \, dA} \right] y \, dA \right] \]

\[ \frac{\text{Ratio of functions of } s}{\text{RATIO OF FUNCTIONS OF } s} \]

Now:

\[ V_0 = \int q \, r ds \]

By substituting for \( q \) and \( q_0 \),

\[ e = \frac{zA}{I} \left[ \int \left[ \frac{\int_0^s \frac{ds}{gt}}{\int y \, dA} \right] y \, dA \right] \]

See example problem
IF SECTION IS SYMMETRIC, $\theta = 0 \implies \delta = 0$

\[ y \perp z = \text{PRINCIPLE AXIS} \]

\[ \text{EM} = 0 = JR' + Fr'' \]

DOUBLE CELL BOX
\[ \phi = \frac{1}{2AG} \oint \frac{q ds}{t} \]

\[ \phi_1 = \frac{1}{2AG} \oint \frac{q_1 ds}{t} \]

\[ \phi_2 = \frac{1}{2AG} \left[ \oint \frac{q_2 ds}{t} - \int_A^B \frac{q_2 ds}{t} \right] \]

\[ \phi = \frac{1}{2AG} \left[ \oint \frac{q_1 ds}{t} + \int_A^B \frac{q_2 ds}{t} \right] \]

\[ M = 2A_1 \phi_1 + 2A_2 \phi_2 \]

\[ \phi_1 = \phi_2 \]

Let

\[ p = \oint \frac{ds}{t} \]

\[ p_1 = \oint_1 \frac{ds}{t} \]

\[ p_2 = \oint_2 \frac{ds}{t} \]

\[ p_1 = \int_A^B \frac{ds}{t} \]

\[ 2A_1 \phi_1 = p_1 q_1 - p_2 q_2 \]

\[ 2A_2 \phi_2 = p_2 q_2 - p_1 q_1 \]

\[ \phi_1 = zG \phi \frac{A_1 p_2 + A_2 p_1}{p_1 p_2 - p_2^2} \]

\[ \phi_2 = zG \phi \frac{A_2 p_1 + A_1 p_2}{p_1 p_2 - p_2^2} \]

REFERENCE: KOLLBRUNNER & BASLER, "TORSION IN STRUCTURES"
SPRINGER-VERLAG, 1969
Torsion of closed sections:

Open section  Closed section  Solid

Solids are used by mechanical engs. can have closed sections with holes or even open sections attached.

Closed sections (thin walled):

Assume there is no tapering along the length (i.e., a cylinder, but not necessarily circular).
\[ \varepsilon \frac{dx}{ds} = 0 = (\frac{dx}{ds}) + \frac{d\mu}{ds} ds - \chi dx \frac{dy}{dx} dx ds + (y + \frac{\partial y}{\partial s} ds) \frac{dy}{ds} \]

\[ \frac{d\mu}{ds} dx ds = \frac{dy}{ds} ds dx \]

\[ \frac{d\lambda}{dx} = \frac{dy}{ds} \]

\[ \varepsilon \frac{dN}{ds} = (N_0 + \frac{dN_0}{ds}) dx - N_0 dx \frac{dy}{dx} ds - (y + \frac{dy}{ds} dx) ds \]

\[ \frac{dN}{ds} ds dx = \frac{dy}{dx} dx ds = 0 \]

\[ \frac{dN}{ds} = \frac{dy}{dx} \]

**But if torsion only \( p = 0 \)**

\[ \therefore N_0 + pp = 0 \Rightarrow N_0 = 0 \]

\[ \therefore \frac{dN_0}{ds} = 0 \]

\[ \therefore \frac{\partial q}{\partial x} = 0 \]

\[ \therefore q = q(x) \]

**From (2)**

\[ \frac{d\mu}{dx} = q(x) \]

\[ \therefore \chi = xq(x) + f(s) \]
IMPOSE BOUNDARY CONDITIONS:

@ \( x = 0 \) \( \frac{N_y}{L} = 0 \) \( \therefore f(s) = 0 \)

@ \( x = L \) \( \frac{N_y}{L} = 0 \) \( \therefore \eta'(s) = 0 \)

\( \therefore \eta'(s) = 0 \)

\( \therefore \eta = \text{constant} \)

SHEAR FLOW, \( \eta = \text{constant} \)

\( \tau = \frac{q}{t} \)

THIS IS FOR ALL THIN WALL, CLOSED CYLINDERS.

NOW CONSIDER A CYLINDER WITH A DISCONTINUITY

\( \Sigma F = 0 = N_1 \sin \alpha \)

\( N_1 = 0 \)

\( \Sigma F_y = 0 = N_2 \cos \alpha - N_0 \)

\( N_0 = 0 \) (UNLESS THE CORNER IS LOADED)
$q = \text{constant}$

And shape of cylinder does not affect results.

Now we have from statics only:

No taper.

$\epsilon = f(s)$

$N_x = 0$ (no axial)

$q = \text{constant} \ (\text{shear flow})$
STATICALLY DETERMINANT: \(N_x, N_0, q\)

Saying:

\[
\frac{\text{MEMBRANE STRENGTH}}{\text{BENDING STRENGTH}} \geq \frac{1000}{1}
\]

\(= \text{NO OUT OF PLANE FORCES}\)
To get a relationship between $M$ and $q$

\[ dA = \frac{rds}{2} \]

Now: \[ M = \oint (q\,ds) \cdot r \] \[ M = q \oint r\,ds \] \[ M = q(2A) \]

\[ \therefore q = \frac{M}{2A} \]

To get a relationship of $M$ and $\phi$ (rotation)

\[ \phi = \frac{\Theta}{L} = \frac{\text{rotation of one end with respect to other}}{\text{length}} \]

= twist

$u =$ displacement in $x$-direction

\[ R\phi x + R\phi dx \]

\[ \frac{d\phi}{dx} \]

\[ R\phi x \]

\[ \frac{du}{ds} ds \]

\[ u + \frac{du}{ds} ds \]
Center of rotation has no translation

\[ r \phi x \]

\[ r \phi x \]

\[ \therefore \gamma = r \phi + \frac{\partial u}{\partial s} \]

In this case we can use \( \frac{\gamma}{s} \) rather than \( \frac{\partial}{\partial s} \)

Because

\[ N_x = 0 \]
\[ \sigma_x = 0 \]
\[ \epsilon_x = 0 \]

\[ \therefore u = f(s) \quad \frac{\partial u}{\partial s} = 0 \]

\[ \tau = \frac{q t}{t} = G \gamma \]

\[ G = \frac{q t}{\gamma} \]

Used to determine effective shear modulus for composites

\[ \gamma = \frac{q t}{G} \]

\[ \frac{q t}{G} = r \phi + \frac{\partial u}{\partial s} \]

\[ du + r \phi ds = \frac{q t}{G} ds \]

\[ \int_{u_0}^{u} du + \frac{\partial}{\partial s} r \phi ds = \frac{q t}{G} \int ds \]

\[ u - u_0 + \frac{\partial}{\partial s} r \phi ds = \frac{q t}{G} \int ds \]

But \( \frac{\partial}{\partial s} r \phi ds = 2A \)

\[ \phi 2A = g \int ds \]

\[ \int ds \]

\[ 5 \]
\[ \phi = \frac{q}{2A} \int \frac{ds}{Gt} \]

\[ \phi = \frac{M}{4A^2} \int \frac{ds}{Gt} \]

\[ \theta = \frac{ML}{4A^2} \int \frac{ds}{Gt} \]

If constant \( G \) and \( t \) then:

\[ \theta = \frac{ML}{4A^2} \text{ (perimeter)} \]

For torsion, \( M = K \theta \)

\[ \therefore K = \frac{M}{\theta} \]

\[ K = \frac{4MA^2}{ML \int \frac{ds}{Gt}} \]

\[ K = \frac{4A^2}{L \int \frac{ds}{Gt}} \text{ Torsional Stiffness} \]
To find warping displacement:

\[ \int_{u_0}^{u_1} d\zeta + \int_{\Omega_0} \gamma \cdot \varphi d\Omega = \int_{\Omega_0} q d\Omega / \gamma \]

\[ u_1 - u_0 = \int_{\Omega_0} q d\Omega / \gamma - \int_{\Omega_0} \gamma \gamma d\Omega \]

Note: This says that the cross section does not remain plane.

If \( q \) was not uniform:

\[ \varphi = \frac{1}{2A} \int_{\Omega} q d\Omega / \gamma \]

This equation can be used to find the shear center.

\[ \varphi = 0 \]
\[ dA = \frac{r ds}{2} \]

\[ dA = \frac{r ds}{2} \]

\[ dA = -\frac{r ds}{2} \]

\[ A = A_1 - A_2 \quad \text{DUE TO DIRECTION CHANGE FOR SHEAR FLOW} \]

\[ \text{IF } A_1 = A_2 \quad \text{THEN } A = 0 \]

\[ \phi = \infty \]

\[ q = \infty \]

\[ \text{SECTION CAN NOT CARRY TORSION} \]
For multiple closed sections:

Now if there was a diaphragm on the end of the section, then \( \phi_A = \phi_B \)

\[
\phi = \frac{M}{JG} \quad J = \text{torsional stiffness}
\]

\[
M_A = \frac{J_A}{J_A + J_B} M
\]
HOW IS THE MOMENT APPLIED?

MUST HAVE A DIAPHRAM TO PRODUCE N CONSTANT q.

\[ M = Fa \]

\[ M = (qa)b + (q\overline{b})a \]

\[ M = 2abq. \]

\[ M = 2Aq. \]

CRANE BOOM

\[ \frac{q}{2} = \frac{Fq}{2ab} \]

\[ M = Fa \]
Diagonal makes end into a diaphragm

\[ P_1 = \left( \frac{F}{2b} \right) b = F/2 \]

\[ P_2 = \left( \frac{F}{2b} \right) a \]

Determine the axial force in diagonal.

There must be a diaphragm everywhere that torque is applied.

Transmission tower

\[ N_x \] does exist in this case therefore the torque causes a shortening of the tower.

By reciprocal theorem, axial forces cause torsion. Diagonal pattern may cause problems for bridges.
Use superposition

\[ \sqrt{10} \]

\[ = \]

\[ 5 \]

\[ + \]

\[ 5 \]

\[ = \]

\[ 5 \]

\[ + \]

\[ 5 \]
NO-DIAPHRAGM OR WEAK DIAPHRAGM.

\[ F \]

\[ M = FL. \]

(FOUNDATION HAS A BIMOMENT LOADING)