Strain Energy

Strain energy is the stored energy in a body due to the applied loads.

For ease of visualization, consider a 1-D loading.

Consider stress moving through distance at different points, some do more work than others.

Total work = \( dW = dU = \int_0^L \int_0^\infty \frac{1}{2} \sigma_x d(\frac{\partial u}{\partial x} dx) dy dz \)

\( W_{\text{external}} \)

Strain energy = \( \int_0^L \int_0^\infty \sigma_x (\varepsilon_x) dx dy dz \)

Differentiating per unit vol

\( U_0 = \int_0^L \sigma_x d\varepsilon_x \)

Hooke's Law

\( U_0 = \int_0^L \varepsilon_x E \varepsilon_x dx \)
Upon integrating and using Hooke's Law

\[ U_0 = \frac{1}{2} E \varepsilon_x^2 \]

\[ = \frac{1}{2} \sigma \varepsilon_x \]

\[ = \frac{1}{2E} \sigma_x^2 \]

\[ U^* = \int_0^{\sigma_x} \varepsilon_x \, d\varepsilon_x \]

\[ U^* = U_0 \text{ for linear-elastic materials} \]
FOR 3-D

\[ U_0 = \frac{1}{2} \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \frac{1}{2} \sigma_z \varepsilon_z \right) \]

FOR SHEAR STRAINS, CONSIDER
Stress \( \sigma_{xy} \) is applied.

The unit force is then \( \sigma \times \text{area} \)

\[ V = \sigma_{xy} dy dz \]

The associated displacement is

\[ \delta_{xy} dy \]

The strain energy is then

\[ V \delta = \frac{1}{2} \left( \sigma_{xy} dx dz \right) \left( \delta_{xy} dy \right) \]

Differentiating with respect to volume

\[ V_0 = \frac{1}{2} \sigma_{xy} \delta_{xy} \]

\[ = \frac{1}{2} \sigma_{xy} \frac{\sigma_{xy}}{2G} \]

\[ = \frac{1}{2} G \delta_{xy}^2 \]

Hooke's law

Work done by shear stress on perpendicular directions = 0

\( \sigma \) Using superposition 3-D strain energy due to shear is

\[ V_0 = \frac{1}{2} \left( \sigma_{xy} \delta_{xy} + \sigma_{xz} \delta_{xz} + \sigma_{yz} \delta_{yz} \right) \]

\( \delta \) General unit strain energy by superposition is

\[ V_0 = \frac{1}{2} \left( \sigma_{xx} \delta_x + \sigma_{yy} \delta_y + \sigma_{zz} \delta_z + \sigma_{xy} \delta_{xy} + \sigma_{xz} \delta_{xz} + \sigma_{yz} \delta_{yz} \right) \]
IMPOSING HOOKE'S LAW RESULTS IN

\[ U_0 = \frac{1}{2} \left( \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \right) - \frac{1}{2} \left( \epsilon_{xx} \sigma_x + \epsilon_{yy} \sigma_y + \epsilon_{zz} \sigma_z \right) \\
+ \frac{1}{2G} \left( \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2 \right) \]

FOR STRESS

FOR STRAIN:

\[ U_0 = \frac{1}{2} \left[ \lambda \epsilon_x^2 + 2 \mu (\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2) + \mu (\epsilon_{xy}^2 + \epsilon_{xz}^2 + \epsilon_{yz}^2) \right] \]

WHERE

\[ \epsilon = \epsilon_x + \epsilon_y + \epsilon_z \quad or \quad \epsilon = \frac{1-2v}{E} (\sigma_x + \sigma_y + \sigma_z) \]

\[ \lambda = \frac{E}{(1+v)(1-2v)} \]

TO SHOW THAT THE ELASTIC CONSTANT MATRIX OF THE GENERALIZED HOOKE'S LAW IS SYMMETRIC, DIFFERENTIATE EQ A W.R.T. STRESS AND EQ B W.R.T. STRAIN

\[ \frac{\partial U_0 (\sigma)}{\partial \sigma_{ij}} = \epsilon_{ij} \]

AND

\[ \frac{\partial U_0 (\epsilon)}{\partial \epsilon_{ij}} = \sigma_{ij} \]

WHERE \( \epsilon_{ij} = \epsilon_{xj}, \epsilon_{ij}, \epsilon_{xz} \)

(NEED TO USE HOOKE'S LAW AS WELL)

LOOK AT EQ. BOTTOM OF PREVIOUS PAGE)
\[ \frac{\partial u}{\partial x} = \sigma_x = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \varepsilon_z \ldots \]

Differentiate again
\[ \frac{\partial^2 u}{\partial x \partial y} = c_{12} = c_{21} \quad \text{etc.} \]

Does not matter which order the differentiation occurs.
Given the rectangular "thin" plate shown, verify that the given stress distribution is valid. 1) Obtain the corresponding stress function, and 2) Determine the resultant normal and shearing boundary forces along the edges $y = 0$ and $x = 6$.

Stress distribution:

$$
\sigma_x = pyx^3 - 2C_1xy + C_2 y
$$

$$
\sigma_y = px^3 - 2px^3y
$$

$$
\tau_{xy} = -\frac{3}{2} px^2y^2 + C_1y^2 + \frac{1}{2} px^4 + C_3
$$

Start with

1) Integration of the differential equations of equilibrium.

$$
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0
$$

$$
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0
$$

$z$-direction is redundant.
2) Imposer Compatibility (Plane Stress)

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = -(1+\nu) \left( \frac{\partial \epsilon_x}{\partial x} + \frac{\partial \epsilon_y}{\partial y} \right)
\]

3) Imposer Boundary Conditions

\begin{align*}
\tau_x &= \sigma_x l + \sigma_{xy} m \\
\tau_y &= \sigma_{xy} l + \sigma_y m
\end{align*}

Surface forces only in \( x \& y \) TERMS

Now, assume body forces are negligible

\[
\frac{\partial \epsilon_x}{\partial x} + \frac{\partial \epsilon_{xy}}{\partial y} = 0
\]

\[
\frac{\partial \epsilon_y}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial x} = 0
\]

\[(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) (\sigma_x + \sigma_y) = 0\]

Assume equilibrium satisfied by a function \( \Phi(x,y) \) (Airy's stress function)

\[
\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}
\]

\[
\sigma_y = \frac{\partial^2 \Phi}{\partial x^2}
\]

\[
\tau_{xy} = \frac{\partial^2 \Phi}{\partial x \partial y}
\]
Now, substitute this back into the equation for compatibility

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \right) = 0 \]

\[ \nabla^2 \phi = 0 \]

\[ \nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^2 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \]

bi-harmonic equation

So now 2-D problem is formulated as a solution of the bi-harmonic equation which must satisfy the boundary conditions.

However, the solution for plane stress is not exact.

Note: \( \varepsilon_x = \varepsilon_y = \varepsilon_{yz} = 0 \)

Also, since \( \sigma_x, \sigma_y, \sigma_{xy} \neq f(\varepsilon) \)

\[ \varepsilon_{xz} = \varepsilon_{yz} = 0 \text{ and} \]

\( \varepsilon_x, \varepsilon_y, \varepsilon_z \text{ and } \sigma_{xy} \neq f(\varepsilon) \)

1. The following compatibility equations must hold

\[ \frac{\partial^2 \varepsilon_x}{\partial x^2} = 0 \quad \frac{\partial^2 \varepsilon_y}{\partial y^2} = 0 \]

\[ \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = 0 \]
If these are imposed on $\nabla^4 \Phi$ it is not satisfied!!!

$\therefore \nabla^4 \Phi$ is an approximation for plane stress, but it is a good approximation for thin plates and shells.

So for problem:

\[
\frac{\partial^4 \Phi}{\partial x^4} = -12 \rho xy
\]

\[
\frac{\partial^4 \Phi}{\partial y^4} = 0
\]

\[
\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 6 \rho xy
\]

\[
\nabla^4 \Phi = -12 \rho xy + 2(6 \rho xy) = 0
\]

The stress field is a possible solution.

2) \[
\frac{\partial^2 \Phi}{\partial x^2} = \rho x y^3 - 2 \rho x^3 y = \sigma_y
\]

Integrating twice

\[
\Phi = \frac{\rho x^3 y^3}{6} - \frac{\rho x^5 y}{10} + f_1(y) x + f_2(y)
\]

Substitute this function into $\nabla^4 \Phi = 0$

\[
\frac{\partial^4 f_1(y)}{\partial y^4} x + \frac{\partial^4 f_2(y)}{\partial y^4} = 0
\]
This is possible if
\[ \frac{d^2 f_1(y)}{dy^2} = 0 \quad \text{and} \quad \frac{d^2 f_2(y)}{dy^2} = 0 \]

A solution is
\[ f_1 = c_4 y^3 + c_5 y^2 + c_6 y + c_7 \]
\[ f_2 = c_8 y^3 + c_9 y^2 + c_{10} y + c_{11} \]

Therefore the stress function is
\[ \Phi = \frac{p x^3 y^3}{6} - \frac{p x^5 y}{10} + (c_4 y^3 + c_5 y^2 + c_6 y + c_7) x + c_8 y^3 + c_9 y^2 + c_{10} y + c_{11} \]

\( c_i \)’s depend on B.C.’s of problem.

3). At \( y = 0 \)
\[ N_x = \int_0^a 2 xy \, dx = \int_0^a \frac{3^2 \Phi}{3x \, dy} \, dx \]
\[ N_x = \frac{pa^5}{5} + 2c_3 at \]

\[ P_y = \int_0^a y \, dx = \int_0^a \frac{2^2 \Phi}{2x^2} \, dx \]
\[ P_y = \int_0^a (p x y^3 + 2 p x^3 y) \, dx = (0) \int_0^a dx = 0 \]
At \( y = b \)

\[
V_x = \int_{-a}^{a} \frac{\partial V}{\partial y} \, dx = \int_{-a}^{a} \frac{\partial^2 \Phi}{\partial x^2} \, dx
\]

\[
V_x = \int_{-a}^{a} \left( -\frac{3}{2} px^2 + c_1 x^2 + \frac{f_x}{2} + c_3 \right) \, dx
\]

\[
\left. V_x \right|_{y=b} = -pa^3 \left( b^2 - \frac{a^2}{2} \right) t + 2a \left( c_1 b^2 + c_3 \right) t
\]

\[
F_x = \int_{-a}^{a} \frac{\partial F}{\partial y} \, dx = \int_{-a}^{a} \frac{\partial^2 \Phi}{\partial y^2} \, dx
\]

\[
F_x = \int_{-a}^{a} \left( py^3 - 2px^3 - 2px^3 b \right) \, dx
\]

\[
\left. F_x \right|_{y=b} = 0\quad \text{t dx} = 0
\]

**Now:** If the problem were to be generalized to a cylinder.

**Ends are free to expand**
ASSUME $\varepsilon_z = $ CONSTANT

Now

$\varepsilon_x = \frac{1 - \nu}{\varepsilon} (\sigma_x - \frac{2\nu}{1-\nu} \sigma_y) - \nu \varepsilon_x$

$\varepsilon_y = \frac{1 - \nu}{\varepsilon} (\sigma_y - \frac{\nu}{1-\nu} \sigma_x) - \nu \varepsilon_y$

$\varepsilon_{xy} = \frac{\varepsilon_{xy}}{\varepsilon}$

$\varepsilon_z = \nu (\sigma_x + \sigma_y) + \varepsilon_z$

IF THE STRAIN EQUATIONS ARE SUBSTITUTED INTO THE EQUATION FOR COMPATIBILITY

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

WE GET

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + 2 \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial y \partial x} = 0$$

WHEN $\sigma_x \neq \sigma_y$ ARE DETERMINED

$$\varepsilon_z = \iint \varepsilon_z \, dx \, dy = 0$$

FOR WHEN $\rho = 0$ (A PIPELINE)
Now that we know the strain energy density (strain per unit volume), we can determine the strain energy in the entire body by integrating over the entire volume.

\[ U = \int_0^V \int_0^H \int_0^W U_0 \, dx \, dy \, dz \]

This is the measure of the body's ability to absorb energy. The higher the value, the better the performance under dynamic loads.

This is not a measure of strength, rather it is a measure of ability to deform, but not yield, which is beneficial for vibration, etc., eliminates (minimizes) fatigue concerns.

Strength/failure depend on geometry.

**Note**: Strain energy is non-linear. It is defined by a quadratic function of either stress or strain. The principle of superposition is not applicable.
STRAIN ENERGY FOR AXIALLY LOADED BARS:

Consider regular general bars. (i.e. non-prismatic.)

\[ \sigma = \frac{P}{A} \]

\( A \) is a function of \( x \)

\[ dU = A \sigma dx \]

\[ U = \int_{x_0}^{x} A \sigma \, dx = \int_{x_0}^{x} \frac{P^2}{2AE} \, dx \]

\[ U = \int_{0}^{L} \frac{P^2}{2AE} \, dx \] \text{only dependent on } E

\[ P = f(x) \quad A = f(x) \]
If the bar is prismatic and the load is only applied to the ends of the bar, the equation can be integrated to obtain

\[ U = \frac{P^2 L}{2AE} \]

Example: (Example 2.7 Text)

At any x-section, the load acting on the x-section is

\[ P = PA(L-x) + P_0 \]

Density \( \rho \)

Area \( A \)

Length of bar supported by x-section

Set up the strain energy integral

\[ U = \int_0^L \frac{PA(L-x) + P_0^2}{2AE} \, dx \]

\[ = \frac{PAL^3}{6E} + \frac{PBL^2}{2E} + \frac{P_0^2 L}{2AE} \]

Self weight

Concentrated load

Additional strain energy due to combined loading
EXAMPLE: Consider previous example with load applied at mid-height.

\[ P = P_m (L-x) + P_0 \]

\[ U = \int_0^{L/2} \frac{[PA(L-x)+P_0]^2}{2AE} \, dx + \int_{L/2}^L \frac{[PA(L-x)]^2}{2AE} \, dx \]

\[ (PA - PAX)^2 = PA^2 - 2PA^2x + PAX^2 \]

\[ \left[ (PA - PAX)^2 \right] = PA^2 - 2PA^2x + PAX^2 - PAXP_0 + PA^2P_0 - PAXP_0 + P_0^2 \]

\[ = PA^2 - 2PA^2x + PAX^2 - 2PAXP_0 + 2PA^2P_0 + P_0^2 \]

\[ U_1 = \int_0^{L/2} \left[ \frac{PA^2}{2E} - \frac{PAX}{2E} + \frac{PAX^2}{2E} - \frac{PAXP_0}{E} + \frac{P_0^2}{2AE} \right] \, dx \]

\[ U_1 = \int_{L/2}^L \left[ \frac{PA^2}{2E} - \frac{PAX}{2E} + \frac{PAX^2}{2E} \right] \, dx \]

\[ U_1 = \left[ \frac{PA^2}{2E} x - \frac{PAX^2}{4E} + \frac{PAX^3}{6E} - \frac{PAXP_0 x^2}{2E} + \frac{P_0^2 x^3}{3AE} \right]_0^{L/2} + \left[ \frac{PA^2}{2E} x - \frac{PAX^2}{2E} x^2 + \frac{PAX^3}{6E} \right]_0^{L/2} \]
\[ U = \frac{\rho^2 AL^3}{4E} - \frac{\rho^2 AL^3}{4E} + \frac{\rho^2 AL^3}{4E} - \frac{\rho^2 AL^3}{4E} + \frac{\rho^2 AL^3}{4E} - \frac{\rho^2 AL^3}{4E} + \frac{P L^2 P_o}{2E} + \frac{P L^2 P_o}{2E} + \frac{P L^2 P_o}{2E} - \frac{P L^2 P_o}{2E} + \frac{P L^2 P_o}{2E} - \frac{P L^2 P_o}{2E} \]

\[ U = \frac{\rho^2 AL^3}{96E} + \frac{3P L^2 P_o}{4E} + \frac{P L^2 P_o}{4AE} \]

\[ \text{Point Load} \]

\[ \text{Additional Energy} \]

\[ \text{Own Weight} \]

\[ \text{Point Load} \]

\[ \frac{\rho L}{4} \]
Strain Energy of Circular Bars in Torsion (shafts)
Consider a bar with circular x-section, varying torque, and varying diameter along the length.

This is a pure shear problem from Table 1.1

\[ T = \frac{T_{T}}{J} \quad \text{@ distance } r \text{ from centroid} \]

\[ U_0 = \frac{1}{2G} \theta^2 \]

\[ = \frac{T_{T}^2}{2GJ^2} \] (EQ 2.40)

Then: (EQ 2.47)

\[ U = \int_0^L \frac{T_{T}^2}{2J^2G} \left( \int r^2 dA \right) dx \]

\[ U_0 \quad \theta \]

\[ U_0 \quad \theta \]

\[ \text{But: } \int r^2 dA = J \]
\[ U = \int_0^L \frac{T^2}{2JG} \, dx \]

For uniform (prismatic) shafts

\[ J = \text{constant} \]

\[ U = \frac{T^2}{2JG} \]

Strain energy for beams

Pure bending (i.e., couple moment)

\[ \sigma = -\frac{My}{I} \]

\[ U_0 = \frac{M^2y^2}{2EI} \]

\[ U = \int_0^L U_0 \, dx = \int_0^L U_0 I \, dx \]

\[ U = \int_0^L \frac{M^2y^2}{2EI} \, dx = \int_0^L \frac{M^2}{2EI} (\int y^2 \, dA) \, dx \]

\[ U = \int_0^L \frac{M^2}{2EI} \, dx \]

\[ U = \int_0^L \frac{M^2}{2EI} \, dx \]
For prismatic beams & uniform moment
C. E. center portion of 4-pt loaded beam or APA's panel quality control test

\[ U = \frac{ML}{2EI} \]

Section under constant moment

Component of strain energy

State of stress:

\[
\begin{bmatrix}
\sigma_m & 0 & 0 \\\n0 & \sigma_m & 0 \\
0 & 0 & \sigma_m
\end{bmatrix}
\]

Volumetric or dilatation

The associated strain

\[ \varepsilon_m = \frac{1}{3} (\varepsilon_x + \varepsilon_y + \varepsilon_z) \]

And

\[ \varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z \]
\[ U_{av} = \frac{3}{2} \sigma_m \leq \sigma_m = \frac{\sigma_m^2}{2K} \]

\[ U_{ov} = \frac{1}{2K} \left( \frac{\sigma_x + \sigma_y + \sigma_z}{3} \right)^2 \]

\[ U_{ov} = \frac{1}{18K} \left( \sigma_x + \sigma_y + \sigma_z \right)^2 \]

\[ K = \frac{E}{3(1-\nu)^2} \] (EQR 2.33 TEXT)

**DEVIATORIC STRESS STATE:**
(DISTORTIONAL)

\[
\begin{bmatrix}
\sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m
\end{bmatrix}
\]

REMEMBER \( (\epsilon_x - \epsilon_m) + (\epsilon_y - \epsilon_m) + (\epsilon_z - \epsilon_m) = 0 \)

REMEMBER (EQR 2.42)

\[ U_0 = \frac{1}{2} \left( \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 \right) - \frac{1}{2} \left( \sigma_{xy} \epsilon_{xy} + \sigma_{yz} \epsilon_{yz} + \sigma_{xz} \epsilon_{xz} \right) + \]
\[ \frac{\nu}{2} \left( \epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{xz}^2 \right) \]

\[ U_{ov} = \frac{1}{6 \varepsilon} \left( \sigma_x + \sigma_y + \sigma_z \right)^2 \]

\[ U_{vo} = U_0 - U_{ov} \] (EXPAND \( U_{ov} \); SUBTRACT FROM \( U_0 \); GROUP TERMS & SIMPLIFY)

\[ U_{vo} = \frac{1}{84} \sqrt{\left( (\sigma_x - \sigma_m)^2 + (\sigma_y - \sigma_m)^2 + (\sigma_z - \sigma_m)^2 \right) + 6 (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2)} \]
IF \( z_{out} = \frac{1}{3} \sqrt{J} \)

THEN \( U_{OD} = \frac{3}{4G} \frac{z^2}{z_{out}} \)

EXAMPLE: (2.8 TEXT)

\[ \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 \\
\end{bmatrix} \]

\( U = 0.25 \)

STATE OF STRESS
(VOLUMETRIC)

\[ \sigma_m = \frac{\sigma}{3} = \frac{P}{3A} \]

DEVIATORIC
\[ \sigma_x - \sigma_m = \frac{2\sigma}{3} = \frac{2P}{3A} \]
\[ \sigma_y - \sigma_m = -\frac{\sigma}{3} = -\frac{P}{3A} \]
\[ \sigma_z - \sigma_m = -\frac{\sigma}{3} = -\frac{P}{3A} \]

\[ \begin{bmatrix}
  \frac{2P}{3A} & 0 & 0 \\
  0 & -\frac{P}{3A} & 0 \\
  0 & 0 & -\frac{P}{3A} \\
\end{bmatrix} \]
\[ U_0 = \frac{1}{2E} \sigma^2 \]  
\[ (\text{Eq} \ 2.38) \]

\[ U_{ov} = \frac{1-2\nu}{6E} \left( \sigma_x + \sigma_y + \sigma_z \right)^2 
= \frac{1-2\nu}{6E} \sigma^2 = \frac{0.5\sigma^2}{6E} = \frac{\sigma^2}{12E} \]

\[ U_{OD} = \frac{3}{4G} \left[ \frac{\sigma^2}{2E} \right] = \frac{1}{12G} \left[ \sigma^2 + \sigma_z^2 \right] \]

\[ U_{OD} = \frac{\sigma_m^2}{E} \]

But \[ G = \frac{E}{2(1+\nu)} \]

\[ U_{OD} = \frac{(1+\nu)\sigma^2}{3E} = \frac{5\sigma^2}{12E} \]

Check \[ U_0 = U_{ov} + U_{OD} = \frac{\sigma^2}{12E} + \frac{5\sigma^2}{12E} = \frac{6\sigma^2}{12E} \]

SAINT-VENANT'S PRINCIPLE.

ALLOWS EQUIVALENT LOADINGS FOR CALCULATION OF \( \sigma \) \& \( e \).

EQUIVALENT LOADINGS ONLY AFFECT DISTRIBUTION OF STRESS \& STRAIN NEAR LOADING \( \sigma \).
Concentrated load [Ref. 2.10]. The average stress $\sigma_{\text{avg}}$ as given by Eq. (1.7) is also sketched in the diagrams. From these, note that the maximum stress $\sigma_{\text{max}}$ greatly exceeds the average stress near the point of application of the load and diminishes as we move along the vertical center axis of the plate away from an end. At a distance equal to the width of the plate, the stress is nearly uniform.

The foregoing observation also holds true for most stress concentrations and practically any type of loading. Thus, the basic formulas of the mechanics of materials give the stress in a member with high accuracy, provided that the cross section in question is at least a distance $b$ (or $h$) away from any concentrated load or discontinuity of shape. Here $b$ (or $h$) denotes the largest lateral dimension of a member. We note that within this distance the stresses depend on the details of loading, boundary conditions, and geometry of the stress concentrations, as is seen in Chapter 3.

Consider, for example, the substitution of a uniform distribution of stress at the ends of a tensile test specimen for the actual irregular distribution that results from end clamping. If we require the stress in a region away from the ends, the stress variation at the ends need not be of concern, since it does not lead to significant variation in the region of interest. As a further example, according to Saint-Venant's principle, the complex distribution of force supplied by the wall to a cantilever beam (Fig. 2.19a) may be replaced

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Figure 2.18 Stress distribution due to a concentrated load in a rectangular elastic plate confirming the Saint-Venant's principle.
```

**PROBLEMS**

**Secs. 2.1 thru**

2.1. Determine the

Here $c$

2.2. Rectangular

F2.2.1 by

where sides $c$

(c) the

2.3. A displ

where

(2, 0, 1)

2.4. The di

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